**Lecture Note-03**

**Complex Integration**

**Line integral in the complex plane**

Complexdefinite integrals are called (complex) **line integrals**. They are written as

.

Here the **integrand**is integrated over a given curve *C.* This curve *C* in the complex plane is

called the **path of integration**.

If *C* is a **closed path** (one whose terminal point coincides with its initial point),

then it is denoted by .

**Partitioning of path *C*:** If *C* is a combination of *C*1 and *C*2 then, 

We may represent *C* by a parametric representation . That is, . The sense of increasing *t* is called the **positive sense** on *C.*

**Note:** Parametric representation of any curve is not unique.

**Example 1:** Find and sketch the path whose orientation is given by .

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Solution:**  Comparing real and imaginary part, we get , .   |  |  |  |  | | --- | --- | --- | --- | |  | *x* | *y* | (*x,y*) | | 1 | 1 | 3 | (1,3) | | 2 | 2 | 6 | (2,6) |   So, represents the line segment from (1,3) to (2,6) in complex plane. | **C:\Users\aiub\Downloads\A-1.jpg**  **Fig: 1** |

**Example 2:** Find and sketch the path whose orientation is given by .

|  |  |
| --- | --- |
| **Solution:**      Comparing real and imaginary part,  we get , .  So, represents upper semicircle of radius 2 with center (0,0). | **C:\Users\aiub\Desktop\1.png**  **Fig: 2** |

**Example 3:** Sketch and represent the line segment from to parametrically.

|  |  |
| --- | --- |
| **Solution:**  The equation of straight line passing through the points (1,1) to (4,-2) is,  That is,  Let, then where *t* varies from to .  So, the parametric equation of line segment from to is,  , . | C:\Users\aiub\Desktop\3.png  **Fig: 3** |

**Example 4:** Sketch and represent unit circle (counterclockwise) parametrically.

|  |  |
| --- | --- |
| **Solution:**  unit circle (counterclockwise)  That is,  (counterclockwise)  Or, .  Let, and ,  Then where  *t* varies from to .  So, the parametric equation of unit circle (counterclockwise) is,  , . | C:\Users\aiub\Downloads\A-4.jpg  **Fig: 4** |

**Example 5:** Sketch the path C consisting of two line segments, one from *z = 0* to *z = 2* and other from *z = 2* to *z = 3+i* , hence evaluate, if *f(z) =* z2.

|  |  |
| --- | --- |
| **Solution:**  Given, C consists of two line segments, one from  *z = 0* to *z = 2* and other from *z = 2* to *z = 3+i.*  **Along**  Equation of the line, which passes through and, is  [using]  We know,,  and varies from to  = . | *C:\Users\aiub\Downloads\A-5.jpg*  **Fig: 5** |
| **Along**  Equation of the line, which passes through and is,  [using]  We know, and varies from to .  =  Now,  +  =. | |

**Example 6.** Sketch the path C from to along the curve  and hence evaluate, where 

|  |  |
| --- | --- |
| **Solution:**  Given, to and .  and  Now,  and | C:\Users\aiub\Downloads\A-6.jpg  **Fig: 6** |
| Therefore,  . | |

**Example 7:** Sketch the path *C* from  *to*  along the curve  and hence evaluate ,where 

|  |  |
| --- | --- |
| **Solution:**  Given, C is the arc from to along the curve.    and, ,  Now, =+    . | **C:\Users\aiub\Downloads\A-7.jpg**  **Fig: 7** |

**Example 8:** Sketch the path *C*from to along the upper half of the circle  and hence evaluate , where  .

|  |  |
| --- | --- |
| **Solution:**  Given, C is the upper half of the circle  from z = -1 to z = 1.  , , where  varies from to , and =  Now, = . | C:\Users\aiub\Desktop\8.png  **Fig: 8** |

**Matlab command to evaluate line integrals:**

|  |  |
| --- | --- |
| 1. Evaluate , where C is the shortest path from 0 to 1+*2i* along   >> fun=@(z) real(z);  >> q=integral(fun,0,1+2i)  q = 0.5000 + 1.0000i | 1. Evaluate , where C consists of the shortest path from z= 0 to z=1 and then to z=1+*2i*   >> fun=@(z) real(z);  >> q=integral(fun, 0,1+2i,'Waypoints',1)  q = 0.5000 + 2.0000i |
| 1. Evaluate , where C is the line segment from z=2 to z=2+3*i* .   >> fun=@(z) conj(z);  >> q=integral(fun,2,2+3i)  q = 4.5000 + 6.0000i |  |

**Sample Exercise Set on Line Integral: 3**

**Sample Exercise**

1. Find and sketch the path and its orientation. Also classify whether the indicated points are interior, exterior or boundary of the following curves-(vi):

(i) (iii)

(ii)

(iv)

(v) ; (5,1)

(vi)

(vii)

(viii)

2. Sketch and represent them parametrically. Also classify whether the indicated points are interior, exterior or boundary of the following curves-(iii & iv):

(i) Line segment from to , (ii) unit circle: (clockwise)

(iii) (counter clockwise) ; (1,6) (iv) (counter clock wise) ; (1,2)

3. Sketch the path from to and hence evaluate .

4. Sketch the path from to and hence evaluate

5. Sketch the path from to and hence evaluate .

6. , is the shortest path from to .

7. Sketch the path which is the circle and hence evaluate .

* Sketch the corresponding paths and hence evaluate them (8-11):

8. , is the shortest path from to .

9., *C* is the patharound the square with vertices .

10. , is the circle , clockwise

Reference Book: Advanced Engineering Mathematics (10th edition) by Erwin Kreyszig, Herbert Kreyszig, Edward J. Norminton, published by John Wiley & Sons, Inc